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An incompatible Korn inequality with conformal dislocation density

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1 Korn's first inequality for compatible fields

Korn's first inequality estimates the L^p-norms of **all** n^2 partial derivatives $\partial_j u_i$ of a vector field $u \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ by L^p-norms of **only** $\frac{n(n+1)}{2}$ particular linear combinations of these partial derivatives [5]:

$$\|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c\,\|\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}, \tag{1}$$

where $\operatorname{sym} P = \frac{1}{2}(P + P^T)$ and if no boundary conditions are imposed then for all $u \in W^{1,p}(\Omega, \mathbb{R}^n)$:

$$\inf_{A\in\mathfrak{so}(n)} \|\mathrm{D}u - A\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c\|\operatorname{sym}\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})}. \tag{2}$$

There exist many different proofs of this remarkable feature [8]. In the L^2-setting, this inequality plays a fundamental role in existence theorems, for uniqueness and stability considerations and for a priori estimates. The L^p-case was used in fluid mechanics, the Stokes problem, Cosserat-type models [6] and general relativity. Many different generalizations have been given, including the case of non-constant coefficients [13], mixed growth conditions [3] or a geometrically non-linear counterpart [4, 7].

Here, we focus on a generalization towards **incompatible fields** $P \neq \mathrm{D}u$. Then (1) generalizes to arbitrary fields P if one adds a term in $\operatorname{Curl} P$ on the right hand side [17, 9, 10], where $\operatorname{Curl} P$ is the **dislocation density tensor**. It turns out that this result can be refined by a relation involving only the deviatoric parts of $\operatorname{sym} P$ and $\operatorname{Curl} P$ [2, 11, 12], where $\operatorname{dev} P = P - \frac{1}{3}\operatorname{tr}(P)\cdot \mathbb{1}$, and the matrix Curl operation is to be understood as a row-wise application.

In [8], we show that P can be estimated in dimension $n = 3$ in terms of $\operatorname{sym} P$ and $\operatorname{dev}\operatorname{sym}\operatorname{Curl} P$. The difference is that we need to subtract not only constants but also certain affine or quadratic skew-symmetric fields in the kernel of the operator $\operatorname{dev}\operatorname{sym}\operatorname{Curl}$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with Lipschitz boundary $\partial\Omega$ and outward unit normal ν. For all $P \in W_0^{1,p,r}(\operatorname{dev}\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3})$, where*

$$W_0^{1,p,r}(\operatorname{dev}\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}) := \{P \in L^p(\Omega;\mathbb{R}^{3\times 3}) \mid \operatorname{dev}\operatorname{sym}\operatorname{Curl} P \in L^r(\Omega;\mathbb{R}^{3\times 3}),\ \operatorname{dev}\operatorname{sym}(P\times\nu) = 0 \text{ on } \partial\Omega\}$$

and $\quad r \in [1,\infty) \quad$ *with* $\quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{3} \quad$ *or* $\quad r > 1 \quad$ *if* $p = \frac{3}{2}$,

there exists a constant $c = c(p, \Omega, r) > 0$ such that

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leq c\,\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{sym}\operatorname{Curl} P\|_{L^r(\Omega,\mathbb{R}^{3\times 3})}\right). \tag{3}$$

If no boundary conditions are imposed then for all $P \in L^p(\Omega, \mathbb{R}^{3\times 3})$:

$$\inf_{T\in K_{S,dSC}} \|P - T\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} \leq c\,\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times 3})} + \|\operatorname{dev}\operatorname{sym}\operatorname{Curl} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{3\times 3})}\right),$$

where the kernel is given by

$$K_{S,dSC} = \{T : \Omega \to \mathbb{R}^{3\times 3} \mid T(x) = \operatorname{Anti}\left(\widetilde{A}\,x + \beta\,x + b + \langle d, x\rangle\, x - \tfrac{1}{2}d\|x\|^2\right),\ \widetilde{A} \in \mathfrak{so}(3), b, d \in \mathbb{R}^3, \beta \in \mathbb{R}\}.$$

2 Application: relaxed micromorphic model

Korn-type inequalities for incompatible tensor fields were originally motivated by applications in infinitesimal gradient plasticity with plastic spin [14] as well as from the relaxed micromorphic model [1, 6]. The latter is a novel micromorphic framework [15, 16] that allows for the description of **microstructure-related frequency band-gaps** [1]. In statics, the goal is to find the macroscopic displacement $u\colon \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and the non-symmetric micro-distortion field $P\colon \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ minimizing

$$\int_\Omega W(\mathrm{D}u, P, \operatorname{Curl} P) - \langle f, u\rangle\, dx \longrightarrow \min,\ (u, P) \in H^1(\Omega) \times H(\operatorname{Curl}),$$

where the elastic energy W is defined as

$$W = \frac{1}{2}\langle \mathbb{C}_e \operatorname{sym}(\mathrm{D}u - P), \operatorname{sym}(\mathrm{D}u - P)\rangle_{\mathbb{R}^{3\times 3}} + \frac{1}{2}\langle \mathbb{C}_{\text{micro}} \operatorname{sym} P, \operatorname{sym} P\rangle_{\mathbb{R}^{3\times 3}} + \frac{1}{2}\langle \mathbb{C}_c \operatorname{skew}(\mathrm{D}u - P), \operatorname{skew}(\mathrm{D}u - P)\rangle_{\mathbb{R}^{3\times 3}} + \frac{\mu L_c^2}{2}\langle \mathbb{L}\operatorname{Curl} P, \operatorname{Curl} P\rangle_{\mathbb{R}^{3\times 3}}.$$

Here, $\mathbb{C}_e$, $\mathbb{C}_{\text{micro}}$: $\operatorname{Sym}(3)\to\operatorname{Sym}(3)$ are classical positive definite 4^{th} order elasticity tensors, $\mathbb{C}_c\colon \mathfrak{so}(3) \to \mathfrak{so}(3)$ is the 4^{th} order rotational coupling tensor, $L_c \geq 0$ is a characteristic length scale, μ is a typical effective shear modulus and $\mathbb{L}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$. The **Euler-Lagrange equations** read

$$\operatorname{Div}[\mathbb{C}_e \operatorname{sym}(\mathrm{D}u - P) + \mathbb{C}_c \operatorname{skew}(\mathrm{D}u - P)] = f,$$
$$\mathbb{C}_e \operatorname{sym}(\mathrm{D}u - P) + \mathbb{C}_c \operatorname{skew}(\mathrm{D}u - P) - \mathbb{C}_{\text{micro}} \operatorname{sym} P - \mu\, L_c^2 \operatorname{Curl}[\mathbb{L}\operatorname{Curl} P] = 0.$$

In view of the new Korn's inequality Theorem 1, for $\mathbb{C}_c \geq 0$, the problem may be even further "relaxed" by requiring only to control

$$\|\operatorname{dev}\operatorname{sym}\operatorname{Curl} P\|^2.$$

3 Algebraic structures

The matrix representation of the cross product

$$a \times b =: \operatorname{Anti}(a)\, b, \qquad a, b \in \mathbb{R}^3 \tag{4}$$

with $\operatorname{Anti}(a) \in \mathfrak{so}(3)$ allows for a cross product with matrices $P \in \mathbb{R}^{3\times 3}$ via

$$P \times b := P\operatorname{Anti}(b), \qquad b \times P := \operatorname{Anti}(b)\, P, \qquad \operatorname{Curl} P = P\operatorname{Anti}(-\nabla).$$

Room's identity $\operatorname{Anti}(a) \times b = b \otimes a - \langle a, b\rangle \cdot \mathbb{1}$ implies $a \otimes b = L(\operatorname{Anti}(a) \times b)$ and interchanging b by ∇ shows $\mathrm{D}A = L(\operatorname{Curl} A)$ for all skew-symmetric matrix fields A, where L denotes a corresponding linear operator with constant coefficients. This yields **Nye's identity**:

$$\mathrm{D}a = \frac{1}{2}\operatorname{tr}(\operatorname{Curl}\operatorname{Anti}(a))\cdot\mathbb{1} - (\operatorname{Curl}\operatorname{Anti}(a))^T, \tag{5a}$$
$$\operatorname{Curl}\operatorname{Anti}(a) = \operatorname{div} a \cdot \mathbb{1} - (\mathrm{D}a)^T. \tag{5b}$$

4 Conformal maps

Conformal Killing fields (or **infinitesimally conformal maps**) are characterized by the condition $\operatorname{dev}\operatorname{sym}\mathrm{D}u = 0$, whose solutions are certain quadratic polynomials. It follows from Nye's identity (5) that the curvature expression in the Cosserat theory $\|\operatorname{dev}\operatorname{sym}\operatorname{Curl}\operatorname{Anti}(a)\|^2$ can be expressed as $\|\operatorname{dev}\operatorname{sym}\mathrm{D}a\|^2$, where the latter has accordingly been termed "conformal curvature" [6]. Therefore, we call the generalized curvature expression $\|\operatorname{dev}\operatorname{sym}\operatorname{Curl} P\|^2$ **conformally invariant dislocation energy**.

5 Proof of conformal Korn's inequalities

For the proof of Theorem 1 we use the Lions lemma resp. Nečas estimate, the compact embedding $L^p \subset\subset W^{-1,p}$ and the linear combination of derivatives

$$\mathrm{D}^3 A = L(\mathrm{D}^2\operatorname{dev}\operatorname{sym}\operatorname{Curl} A)$$

for skew-symmetric matrix fields $A : \Omega \to \mathfrak{so}(3)$.

Theorem 2 (Lions lemma and Nečas estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $m \in \mathbb{Z}$, $p \in (1,\infty)$ and $f \in \mathscr{D}'(\Omega)$. Then*

$$\mathrm{D}f \in W^{m-1,p} \implies f \in W^{m,p} \quad \text{and} \quad \|f\|_{W^{m,p}} \leq c\,(\|f\|_{W^{m-1,p}} + \|\mathrm{D}f\|_{W^{m-1,p}})$$

6 New Banach spaces

For general matrix fields, $\mathrm{D}\operatorname{Curl} P = L(\mathrm{D}\operatorname{dev}\operatorname{Curl} P)$, cf. [11], which implies that

$$\operatorname{Curl} P \in W^{m,p} \iff \operatorname{dev}\operatorname{Curl} P \in W^{m,p} \quad \forall m \in \mathbb{Z}.$$

However, on bounded, open, non-empty sets $\Omega \subset \mathbb{R}^3$ we have the non-equivalence

$$W^{1,p}(\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}) \subsetneq W^{1,p}(\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}) \subsetneq W^{1,p}(\operatorname{dev}\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{R}^{3\times 3}).$$

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